The Integral Calculus

James Ballantyne

THE INTEGRAL CALCULUS

ON THE INTEGRATION OF

THE POWERS OF TRANSCENDENTAL FUNCTIONS, NEW METHODS AND THEOREMS, CALCULATION OF THE BERNOULLIAN NUMBERS, RECTIFICATION OF THE LOGARITHMIC CURVE, INTEGRATION OF LOGARITHMIC BINOMIALS, ETC.

BY

JAMES BALLANTYNE

PUBLISHED BY THE AUTHOR

6 GREENVILLE ST., BOSTON

1919

THE·PLIMPTON·PRESS
NORWOOD·MASS·U·S·A

CONTENTS

THE INTEGRAL CALCULUS

SECTION ONE

On the Integration of the Powers of Trigonometrical Functions

1. In performing the operation of integrating a differential it is usual to add the constant C to the result, because the integral from which the differential has been derived may contain some constant quantity not affected by the variable; in which case the constant would not be indicated in the differential. In the following Tables and Series the constant has been fully accounted for, and the complete integrals are therein expressed, excepting in those cases where the constant is particularly noted.

TABLE I

Integrals of Even Powers of sin $x \cdot dx$, to Radius 1

$$\int \sin^0 x \cdot dx = x.$$

$$\int \sin^2 x \cdot dx = \frac{x - \cos x \cdot \sin x}{2}.$$

$$\int \sin^4 x \cdot dx = \frac{3x - 3 \cos x \cdot \sin x - 2 \cos x \cdot \sin^3 x}{2 \cdot 4}.$$

$$\int \sin^6 x \cdot dx = \frac{15\, x - 15 \cos x \cdot \sin x - 10 \cos x \cdot \sin^3 x - 8 \cos x \cdot \sin^5 x}{2 \cdot 4 \cdot 6}.$$

$$\int \sin^8 x \cdot dx = (105\, x - 105 \cos x \cdot \sin x - 70 \cos x \cdot \sin^3 x - 56 \cos x \cdot \sin^5 x - 48 \cos x \cdot \sin^7 x) \div 2 \cdot 4 \cdot 6 \cdot 8$$

$$\int \sin^{10} x \cdot dx = (945\, x - 945 \cos x \cdot \sin x - 630 \cos x \cdot \sin^3 x - 504 \cos x \cdot \sin^5 x - 432 \cos x \cdot \sin^7 x - 384 \cos x \cdot \sin^9 x) \div 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10$$

2. This table may be carried to any extent by observing the following law of progression:

$$\int \sin^m x \cdot dx = \frac{m-1}{m} \int \sin^{m-2} x \cdot dx - \frac{1}{m} \cos x \cdot \sin^{m-1} x,$$

and the table may be extended to negative values of m, by changing this expression to

$$\int \sin^{m-2}x \cdot dx = \frac{m}{m-1}\int \sin^m x \cdot dx + \frac{1}{m-1}\cos x \cdot \sin^{m-1}x; \text{ thus,}$$

$$\int \sin^{-2}x \cdot dx = -\cot x + C.$$

$$\int \sin^{-4}x \cdot dx = -\frac{2\cot x + \dfrac{\cot x}{\sin^2 x}}{3} + C.$$

$$\int \sin^{-6}x \cdot dx = -\frac{8\cot x + \dfrac{4\cot x}{\sin^2 x} + \dfrac{3\cot x}{\sin^4 x}}{3\cdot 5} + C.$$

$$\int \sin^{-8}x \cdot dx = -\frac{48\cot x + \dfrac{24\cot x}{\sin^2 x} + \dfrac{18\cot x}{\sin^4 x} + \dfrac{15\cot x}{\sin^6 x}}{3\cdot 5\cdot 7} + C.$$

The value of C is the area of the full quadrant of the curvilinear for each particular value of m; that is,

$$C = \int_0^{\frac{\pi}{2}} \sin^m x \cdot dx.$$

Table II

Integrals of Even Powers of $\cos x \cdot dx$, to Radius 1

$$\int \cos^0 x \cdot dx = x.$$

$$\int \cos^2 x \cdot dx = \frac{x + \sin x \cdot \cos x}{2}.$$

$$\int \cos^4 x \cdot dx = \frac{3x + 3\sin x \cdot \cos x + 2\sin x \cdot \cos^3 x}{2\cdot 4}.$$

$$\int \cos^6 x \cdot dx = \frac{15x + 15\sin x \cdot \cos x + 10\sin x \cdot \cos^3 x + 8\sin x \cdot \cos^5 x}{2\cdot 4\cdot 6}.$$

$$\int \cos^8 x \cdot dx = (105x + 105\sin x \cdot \cos x + 70\sin x \cdot \cos^3 x + 56\sin x \cdot \cos^5 x + 48\sin x \cdot \cos^7 x) \div 2\cdot 4\cdot 6\cdot 8.$$

$$\int \cos^{10} x \cdot dx = (945x + 945\sin x \cdot \cos x + 630\sin x \cdot \cos^3 x + 504\sin x \cdot \cos^5 x + 432\sin x \cdot \cos^7 x + 384\sin x \cdot \cos^9 x) \div 2\cdot 4\cdot 6\cdot 8\cdot 10$$

3. This table may be extended by the following law of progression:

$$\int \cos^m x \cdot dx = \frac{m-1}{m}\int \cos^{m-2}x \cdot dx + \frac{1}{m}\sin x \cdot \cos^{m-1}x,$$

and for negative values of m, this expression may be changed to

$$\int \cos^{m-2}x \cdot dx = \frac{m}{m-1}\int \cos^m x \cdot dx - \frac{1}{m-1}\sin x \cdot \cos^{m-1}x,$$

and from this we get

$$\int \cos^{-2}x \cdot dx = \tan x.$$

$$\int \cos^{-4}x \cdot dx = \frac{2\tan x + \dfrac{\tan x}{\cos^2 x}}{3}.$$

$$\int \cos^{-6}x \cdot dx = \frac{8\tan x + \dfrac{4\tan x}{\cos^2 x} + \dfrac{3\tan x}{\cos^4 x}}{3\cdot 5}.$$

$$\int \cos^{-8}x \cdot dx = \frac{48\tan x + \dfrac{24\tan x}{\cos^2 x} + \dfrac{18\tan x}{\cos^4 x} + \dfrac{15\tan x}{\cos^6 x}}{3\cdot 5\cdot 7}.$$

Table III

Integrals of Odd Negative Powers of Cos $x \cdot dx$, to Radius 1

$$\int \cos^{-1}x \cdot dx = \tfrac{1}{2}\log\frac{1+\sin x}{1-\sin x}.$$

$$\int \cos^{-3}x \cdot dx = \tfrac{1}{2}\log\frac{1+\sin x}{1-\sin x} + \frac{\tan x}{\cos x}.$$

$$\int \cos^{-5}x \cdot dx = \frac{\tfrac{3}{2}\log\dfrac{1+\sin x}{1-\sin x} + \dfrac{3\tan x}{\cos x} + \dfrac{2\tan x}{\cos^3 x}}{2\cdot 4}.$$

$$\int \cos^{-7}x \cdot dx = \frac{\tfrac{15}{2}\log\dfrac{1+\sin x}{1-\sin x} + \dfrac{15\tan x}{\cos x} + \dfrac{10\tan x}{\cos^3 x} + \dfrac{8\tan x}{\cos^5 x}}{2\cdot 4\cdot 6}.$$

The law of progression is

$$\int \frac{1}{\cos^{m+2}x}\cdot dx = \frac{m}{m-1}\int \frac{1}{\cos^m x}\cdot dx + \frac{1}{m+1}\cdot\frac{\tan x}{\cos^m x}.$$

Table IV

Integrals of Odd Negative Powers of Sin $x \cdot dx$, to Radius 1.

$$\int \sin^{-1}x \cdot dx = -\tfrac{1}{2}\log\frac{1+\cos x}{1-\cos x} + C.$$

$$\int \sin^{-3}x \cdot dx = -\frac{\tfrac{1}{2}\log\dfrac{1+\cos x}{1-\cos x} + \dfrac{\cot x}{\sin x}}{2} + C.$$

$$\int \sin^{-5}x \cdot dx = -\frac{\tfrac{3}{2}\log\dfrac{1+\cos x}{1-\cos x} + \dfrac{3\cot x}{\sin x} + \dfrac{2\cot x}{\sin^3 x}}{2\cdot 4} + C.$$

$$\int \sin^{-7}x \cdot dx = -\frac{\tfrac{15}{2}\log\dfrac{1+\cos x}{1-\cos x} + \dfrac{15\cot x}{\sin x} + \dfrac{10\cot x}{\sin^3 x} + \dfrac{8\cot x}{\sin^5 x}}{2\cdot 4\cdot 6} + C.$$

The law of progression is

$$\int \frac{1}{\sin^{m+2}x} \cdot dx = \frac{m}{m+1} \int \frac{1}{\sin^m x} \cdot dx - \frac{1}{m+1} \cdot \frac{\cot x}{\sin^m x}.$$

The value of C is the full quadrant of the curvilinear.

4. The value assigned to C in the integrals of $\frac{1}{\sin^m x} \cdot dx$, in Tables I and IV, is readily derived from the following consideration. If we trace, for the full quadrant, the two curves, $y = \frac{1}{\sin^m x}$ and $y = \frac{1}{\cos^m x}$, having their origin at opposite ends of the axis of x, the two curves will coincide throughout. Therefore, the value of $\int \frac{1}{\sin^m x} \cdot dx$, given by Tables I and IV, simply measures negatively the value given by Tables II and III for $\int \frac{1}{\cos^m x} \cdot dx$. Hence, $\int \frac{1}{\sin^m x} \cdot dx$ measures negatively the complement of the area of the curvilinear; so that, adding to this negative quantity the full quadrant of the curvilinear gives the proper integral. These remarks equally apply to the curves $y = \tan^m x$ and $y = \frac{1}{\tan^m x}$, the integrals of which are given on another page.

Series I

The integral of $\sin^m x \cdot dx$, for any value of m, is $A \cdot (1 - \cos x) - \frac{B}{3}(1 - \cos^3 x) + \frac{C}{5}(1 - \cos^5 x) - \frac{D}{7}(1 - \cos^7 x) + \ldots$ where $A, B, C, D \ldots$ are the successive terms in the development of the binomial $(1 + 1)^{\frac{m-1}{2}}$; namely, $A = 1$; $B = \frac{m-1}{2}$;

$$C = \frac{m - 1 \cdot m - 3}{2 \cdot 4}; \; D = \frac{m - 1 \cdot m - 3 \cdot m - 5}{2 \cdot 4 \cdot 6} \ldots$$

5. Series I terminates with the term containing the mth power of $\cos x$, when m is a positive odd integer; otherwise it is infinite. It therefore gives complete expressions for odd positive powers of $\sin x \cdot dx$.

Series II

The integral of $\cos^m x \cdot dx$, for any value of m, is $A \cdot \sin x - \frac{B}{3} \sin^3 x + \frac{C}{5} \sin^5 x - \frac{D}{7} \sin^7 x + \ldots$ where $A, B, C, D \ldots$ are the successive terms of the binomial $(1 + 1)^{\frac{m-1}{2}}$, as before.

This series will terminate with the mth power of $\sin x$ when m is a positive odd integer; otherwise it is infinite. It, therefore, gives complete expressions for positive odd powers of $\cos x \cdot dx$.

Series III

6. The integral of $\tan^m x \cdot dx$, for even values of m, is $\pm x \mp \tan x \pm \frac{\tan^3 x}{3} \mp \frac{\tan^5 x}{5} \pm \ldots$ to term $\frac{\tan^{m-1} x}{m-1}$, using the upper or lower sign according as $\frac{m}{2}$ is even or odd.

This series, as therein indicated, will terminate with the $(m-1)$th power of $\tan x$.

Series IV

7. The integral of $\tan^m x \cdot dx$, for odd values of m, is $\pm \log \cos x \pm \frac{\tan^2 x}{2} \mp \frac{\tan^4 x}{4} \pm \frac{\tan^6 x}{6} \mp \ldots$ to term $\frac{\tan^{m-1} x}{m-1}$; using the upper or lower sign according as $\frac{m-1}{2}$ is odd or even.

This series, as therein indicated, terminates with the $(m-1)$th power of $\tan x$.

Series V

8. The integral of $\cot^m x \cdot dx$, for even values of m, is

$$\pm x \pm \cot x \mp \frac{\cot^3 x}{3} \pm \frac{\cot^5 x}{5} \mp \ldots \text{ to term } \frac{\cot^{m-1} x}{m-1}, + C,$$

using the upper or lower sign according as $\frac{m}{2}$ is even or odd.

The value of C is the area of the full quadrant of the curvilinear, $\mp \frac{\pi}{2}$.

Series VI

9. The integral of $\cot^m x \cdot dx$, for odd values of m, is $\mp \log \sin x \mp \frac{\cot^2 x}{2} \pm \frac{\cot^4 x}{4} \mp \frac{\cot^6 x}{6} \pm \ldots$ to term $\frac{\cot^{m-1} x}{m-1}$, $+ C$ using the upper or lower sign according as $\frac{m-1}{2}$ is odd or even. C is the full quadrant of the curvilinear. Series V and VI, as therein indicated, terminate with the (m – 1)th power of $\cot x$.

For the value of C, see remarks under Art. 4.

Series VII

10. The integral of $\tan^m x \cdot dx$, for any value of m, is $\frac{\tan^{m+1} x}{m+1} - \frac{\tan^{m+3} x}{m+3} + \frac{\tan^{m+5} x}{m+5} - \frac{\tan^{m+7} x}{m+7} + \ldots$ When m is an even nega-

tive, the constant C to be added is the full quadrant of the curvilinear, $\pm \frac{\pi}{2}$; using the upper or lower sign according as $\frac{m}{2}$ is even or odd. When m is an odd negative, the constant C to be added is the full quadrant of the curvilinear. When m is an odd negative, there will always appear in the series one irrational term, namely, $\pm \frac{1}{0} \tan^0 x$. It is quite permissible to suppose that this term has been obtained by integrating, with respect to the tangent, the quantity $\frac{1}{1} \tan^{-1} x, = \frac{1}{\tan x}$; but the integral of this, with respect to the tangent, is $\log \tan x$. Therefore, $\log \cdot \tan x$ must always be substituted for the irrational term, using the sign $+$ or $-$ according as it is in the series.

11. Series VII is infinite for any single integral; but it terminates for certain pairs of integrals, thus:

$$\int \tan^m x \cdot dx + \int \tan^{m+2} x \cdot dx = \frac{\tan^{m+1} x}{m+1};$$

$$\int \tan^m x \cdot dx + \int \tan^{m+6} x \cdot dx = \frac{\tan^{m+1} x}{m+1} - \frac{\tan^{m+3} x}{m+3} + \frac{\tan^{m+5} x}{m+5};$$

and, r being any whole number divisible by 4,

$$\int \tan^m x \cdot dx + \int \tan^{m+2+r} x \cdot dx = \frac{\tan^{m+1} x}{m+1} - \frac{\tan^{m+3} x}{m+3} + \frac{\tan^{m+5} x}{m+5} - \ldots$$

to term $\pm \frac{\tan^{m+1+r} x}{m+1+r}$.

SECTION TWO

Showing how Some Well-known Mathematical Formulae, derived from other Sources, may be obtained from the Foregoing Integrals

12. It will be observed that each of the integrals given by Tables I and II, for positive values of m, contains x; that is, some definite proportion of π; while π does not enter into the values given by Series I and II. It is, therefore, evident that a great variety of series for expressing π may be had by equating the value given by the series to that given by the tables. A few examples are here given.

Let $x = \frac{\pi}{2}$, and equate Series I to Table I, for

$$m = 0, \text{ then } 1 + \frac{1}{2\cdot 3} + \frac{3}{2\cdot 4\cdot 5} + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 7} + \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 9} + \ldots = \frac{\pi}{2}.$$

$$m = 2, \text{ then } 1 - \frac{1}{2\cdot 3} - \frac{1}{2\cdot 4\cdot 5} - \frac{3}{2\cdot 4\cdot 6\cdot 7} - \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 8\cdot 9} - \ldots = \frac{\pi}{4}.$$

$$m = 4, \text{ then } 1 - \frac{3}{2\cdot 3} + \frac{3}{2\cdot 4\cdot 5} + \frac{3}{2\cdot 4\cdot 6\cdot 7} + \frac{3\cdot 3}{2\cdot 4\cdot 6\cdot 8\cdot 9}$$

$$+ \frac{3\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 11} + \ldots = \frac{3\pi}{16}.$$

13. By equating Series II to Table II for $m = 0$, we get

$$S + \frac{1}{2\cdot 3}S^3 + \frac{3}{2\cdot 4\cdot 5}S^5 + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 7}S^7 + \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 9}S^9 + \ldots = x.$$

This series expresses the arc in terms of the sine; and it is simply an inversion of the series given by Newton for the sine in terms of the arc.

14. By Table I, the integral of $\sin^2 x\cdot dx$ is $\frac{x - \cos x\cdot \sin x}{2}$, and by Series I, it is

$$(1 - c) - \frac{1}{2\cdot 3}(1 - c^3) - \frac{1}{2\cdot 4\cdot 5}(1 - c^5) - \frac{3}{2\cdot 4\cdot 6\cdot 7}(1 - c^7)$$

$- \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 8\cdot 9}(1 - c^9) - \ldots$ or the equivalent series given in Art. 29,

$$\frac{1}{3}S^3 + \frac{1}{2\cdot 5}S^5 + \frac{3}{2\cdot 4\cdot 7}S^7 + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 9}S^9 + \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 11}S^{11} + \ldots$$

These two series express the same quantity as the integral of $\sqrt{2x - x^2}$, in the equation to the circle $y = \sqrt{2x - x^2}$. (Here, x is an algebraic, not a transcendental quantity, although it is incidentally the versed sine of the arc.) This may be shown by tracing the two curves.

FIG. 1.

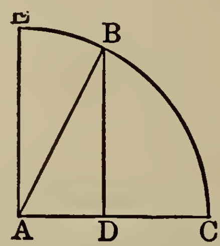


FIG. 2.

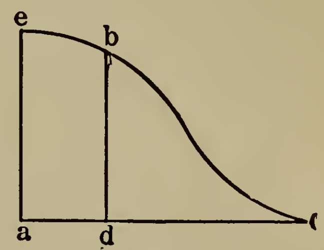


Figure 1 is a quadrant of the circle $y = \sqrt{2x - x^2}$; Figure 2 is a quadrant of the curve $y = \sin^2 x$.

$AC = 1$, $ac = \dfrac{\pi}{2} =$ arc EBC; DC and $cd = x$, BD and $bd = y$ in the respective equations. The areas BCD and bcd are the integrals of y. The area bcd, as shown above, is $\dfrac{x}{2} - \dfrac{\cos x \cdot \sin x}{2}$.

When the arc $BC =$ axis $cd = x$, then

Area $BCD = BCA - BDA = \dfrac{x}{2} - \dfrac{\cos x \cdot \sin x}{2} =$ area bcd.

In Figure 2 the length of the curve for a full quadrant is the same as the length of the curve of sines, and the same as the ellipse whose semi-axes are $\sqrt{2}$ and 1. And certain modifications of the curve of sines, by introducing constants, give the same length of curve as the ellipse whose semiaxes are a and b; thus: $y = \sqrt{a^2 - b^2} \cdot \sin \cdot \dfrac{x}{b}$ gives the same length of curve as the ellipse, with the centre as origin of coördinates; and $y = \sqrt{a^2 - b^2} \cdot \cos \cdot \dfrac{x}{b}$ gives the same as the ellipse, with the apex as origin.

This may be shown geometrically, but the demonstration would be foreign to the purpose of this work.

15. Let the complete integrals of $\sin^m x \cdot dx$ for odd values of m be deduced from Series I, thus:

$$\int \sin x \cdot dx = 1 - \cos x.$$
$$\int \sin^3 x \cdot dx = \tfrac{2}{3} - \cos x + \tfrac{1}{3}\cos^3 x.$$
$$\int \sin^5 x \cdot dx = \tfrac{2}{3}\cdot\tfrac{4}{5} - \cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x.$$

And, if the full quadrant be taken, $x = \dfrac{\pi}{2}$; therefore,

$$\int_0^{\frac{\pi}{2}} \sin^m x \cdot dx = \tfrac{2}{3}\cdot\tfrac{4}{5}\cdot\tfrac{6}{7}\cdot\tfrac{8}{9}\cdot\tfrac{10}{11} \ldots \text{ to } \frac{m-1}{m}.$$

Now, by Table I, for even values of m, we get

$$\int_0^{\frac{\pi}{2}} \sin^m x \cdot dx = \frac{\pi}{2}\left(\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{7}{8}\cdot\tfrac{9}{10}\ \ldots \text{ to } \frac{m-1}{m}\right).$$

But, if m be considered infinite, the distinction between odd and even values of m vanishes, and these two expressions will be equal to each other. Hence, if the first expression be divided by that part of the second which is within the parentheses, we get

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdot\frac{10\cdot 10}{9\cdot 11}\ \ldots,$$

and this is identical with the series given by Wallis for the value of π.

16. If the integrals given by Table II for negative values of m be equated to those given by Series II, we get expressions for the tangent in terms of the sine. Thus, for $\int \cos^{-2} x$,

$$\sin x + \tfrac{1}{2}\sin^3 x + \frac{3}{2\cdot 4}\sin^5 x + \frac{3\cdot 5}{2\cdot 4\cdot 6}\sin^7 x + \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\sin^9 x + \ldots$$
$$= \tan x.$$

And this is the series that results from dividing the sine by the cosine; that is $\dfrac{\sin x}{\sqrt{1-\sin^2 x}}$, the denominator being reduced to a series.

17. For any even value of m, if the integral of $\tan^m x \cdot dx$ given by Series VII be equated to that given by Series III when m is positive, or to that given by Series V when m is negative, the result in each case will be,

$$x = \tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \tfrac{1}{7}\tan^7 x + \ldots$$

and this is Gregory's well-known series for expressing the arc in terms of the tangent.

The separate values of the positive and negative parts of this series may be had by comparing it with the integral of $\dfrac{1}{\cos x}\,dx$, given by Series II, thus;

The positive part $= \frac{1}{4}\log\dfrac{1+\tan x}{1-\tan x} + \frac{1}{2}x.$

The negative part $= \frac{1}{4}\log\dfrac{1+\tan x}{1-\tan x} - \frac{1}{2}x.$

This is true only when the tangent is less than 1, because, when it is 1 or greater than 1, each part is infinite.

18. For any odd value of m, if the integral of $\tan^m x \cdot dx$ given by Series VII be equated to that given by Series IV when m is positive,

or to that given by Series VI when m is negative, the result will be

$$\tfrac{1}{2}\tan^2 x - \tfrac{1}{4}\tan^4 x + \tfrac{1}{6}\tan^6 x - \tfrac{1}{8}\tan^8 x + \ldots = -\log\cos x,$$

or its equivalent,

$$\tfrac{1}{2}\cot^2 x - \tfrac{1}{4}\cot^4 x + \tfrac{1}{6}\cot^6 x - \tfrac{1}{8}\cot^8 x + \ldots = -\log\sin x.$$

Making $\tan x = 1$, and multiplying the result by 2, we get

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \ldots = \log 2.$$

Making $\tan x = \frac{1}{2}$, we get

$$\tfrac{1}{4} - \frac{1}{2(4)^2} + \frac{1}{3(4)^3} - \frac{1}{4(4)^4} + \ldots = \log 5 - \log 4.$$

If a and z be any positive quantities, $\dfrac{a}{a+z}$ will represent a cosine in the first quadrant, and its corresponding tangent will be $\dfrac{\sqrt{2az+z^2}}{a}$. Using this value of the tangent, we get a series for expressing the difference between $\log a$ and $\log(a+z)$;

$$\frac{2az+z^2}{2a^2} - \frac{(2az+z^2)^2}{4a^4} + \frac{(2az+z^2)^3}{6a^6} - \ldots = \log(a+z) - \log a.$$

And, making $z = 1$, and $a =$ the consecutive numbers in successive operations, we get a series for constructing tables of logarithms, thus:

$$\frac{2a+1}{2a^2} - \frac{(2a+1)^2}{4a^4} + \frac{(2a+1)^3}{6a^6} - \frac{(2a+1)^4}{8a^8} + \ldots = \log(a+1) - \log a.$$

Again, making the cosine $\dfrac{\sqrt{a}}{\sqrt{a+z}}$, its tangent will be $\dfrac{\sqrt{z}}{\sqrt{a}}$. Using this value of the tangent, and multiplying the result by 2, we get,

$$\frac{z}{a} - \frac{z^2}{2a^2} + \frac{z^3}{3a^3} - \frac{z^4}{4a^4} + \ldots = \log(a+z) - \log a.$$

And this is a well-known series for constructing tables of logarithms.

In this series, z may be considered negative as well as positive, by placing the tangent in the second quadrant; hence, we get,

$$\frac{z}{a} + \frac{z^2}{2a^2} + \frac{z^3}{3a^3} + \frac{z^4}{4a^4} + \ldots = \log a - \log(a-z),$$

and, taking the sum and difference of these two series, we have

$$2\left(\frac{z}{a} + \frac{z^3}{3a^3} + \frac{z^5}{5a^5} + \frac{z^7}{7a^7} + \ldots\right) = \log(a+z) - \log(a-z);$$

and

$$2\left(\frac{z^2}{2a^2} + \frac{z^4}{4a^4} + \frac{z^6}{6a^6} + \frac{z^8}{8a^8} + \ldots\right) = \log a^2 - \log(a^2 - z^2).$$

19. By Series II, for the integral of $\frac{1}{\cos x}\cdot dx$, we get, $\sin x + \frac{1}{3}\sin^3 x + \frac{1}{5}\sin^5 x + \ldots$ which is the same in form as the series given above for $\log(a+z) - \log(a-z)$, when we make $\sin x = \frac{z}{a}\cdot$ In which case, the sum of this series is $\frac{1}{2}\log(a+z) - \frac{1}{2}\log(a-z) = \frac{1}{2}\log\frac{1+\sin x}{1-\sin x}$; and this agrees with the value of $\int\frac{1}{\cos x}\cdot dx$, given by Table III.

SECTION THREE

On the Series for expressing the Tangent in Terms of the Arc, and its Relation to the Bernoullian Numbers.

20. The application of Maclaurin's theorem of successive differentials, to express the tangent in terms of the arc, is generally considered too troublesome for practical use on account of the complexity of the derived functions. I offer a solution of this problem, in which the difficulty is obviated by applying Maclaurin's theorem to the equation $\int \tan x \cdot dx = -\log \cos x$, given by Series IV.

Assume

$$w = -\log \cos x = Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \dots \qquad (1)$$

$$\frac{dw}{dx} = \tan x = A + 2Bx + 3Cx^2 + 4Dx^3 + \dots \qquad (2)$$

$$\frac{d^2w}{dx^2} = 1 + \tan^2 x = 2B + 2\cdot 3Cx + 3\cdot 4Dx^2 + 4\cdot 5Ex^3 + \dots \text{ hence}$$

$$\tan^2 x = -1 + 2B + 2\cdot 3Cx + 3\cdot 4Dx^2 + 4\cdot 5Ex^3 + \dots \qquad (3)$$

By taking the square of series (2) for $\tan x$, and equating its cœfficients for like powers of x to those of series (3) for $\tan^2 x$, we get, $A, C, E, G, I \dots$ each $= 0$.

$$B = \tfrac{1}{2},\ D = \frac{1}{3\cdot 4},\ F = \frac{1}{5\cdot 9},\ \mathrm{H} = \frac{17}{5\cdot 7\cdot 8\cdot 9},\ J = \frac{62}{5\cdot 7\cdot 9\cdot 9\cdot 10}, \dots$$

Raise these fractions so as to have an obvious progression in the denominators, and substitute them for $B, D, F, H, \dots$ in series (2) and we get,

$$\tan x = x + \frac{2}{2\cdot 3}x^3 + \frac{16}{2\cdot 3\cdot 4\cdot 5}x^5 + \frac{272}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7$$

$$+ \frac{7936}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}x^9 + \dots \qquad (4)$$

Here, the numerators of the cœfficients cannot be determined by inspection, and it required much thought to evolve their law of progression, which is as follows:

21. Let $N_1, N_3, N_5, N_7 \dots$ represent the numerators of the cœfficients of $x, x^3, x^5, x^7 \dots$; and let a represent the index of the power of x whose cœfficient is required; then the numerator, N_a, of the cœfficient x^a is

$$\pm 1 \mp aN_1 \pm \frac{a\cdot a-1\cdot a-2}{2\cdot 3}N_3 \mp \frac{a\cdot a-1\cdot a-2\cdot a-3\cdot a-4}{2\cdot 3\cdot 4\cdot 5}N_5 \pm \dots$$

to the term N_{a-2}. Using the upper or lower sign according as $\frac{a-1}{2}$ is even or odd.

A table of these numerators is given at the end of this section.

If the values of B, D, F, H . . . be applied to series (1) Art. 20, we get

$$-\log\cos x = \tfrac{1}{2}x^2 + \frac{2}{2\cdot 3\cdot 4}x^4 + \frac{16}{2\cdot 3\cdot 4\cdot 5\cdot 6}x^6 + \frac{272}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}x^8 + \ldots$$

which is simply Series (4), Art. 20, integrated with respect to x.

22. The above law for the numerators of coefficients may be modified so as to apply to even powers of x, thus,

$$\pm 1 \mp \frac{a\cdot a-1}{2}N_2 \pm \frac{a\cdot a-1\cdot a-2\cdot a-3}{2\cdot 3\cdot 4}N_4$$

$$\mp \frac{a\cdot a-1\cdot a-2\cdot a-3\cdot a-4\cdot a-5}{2\cdot 3\cdot 4\cdot 5\cdot 6}N_6 \pm \ldots$$

to the term N_{a-2}. Using the upper or lower sign according as $\frac{a}{2}$ is odd or even. From this we get

$$1 + \tfrac{1}{2}x^2 + \frac{5}{2\cdot 3\cdot 4}x^4 + \frac{61}{2\cdot 3\cdot 4\cdot 5\cdot 6}x^6 + \frac{1385}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}x^8$$

$$+ \frac{50521}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}x^{10} + \ldots \tag{1}$$

Now, the square of this series is

$$1 + x^2 + \frac{16}{2\cdot 3\cdot 4}x^4 + \frac{272}{2\cdots 6}x^6 + \frac{7936}{2\cdots 8}x^8 + \ldots \tag{2}$$

But this is the differential with respect to x, of series (4), Art. 20; namely, $1 + \tan^2 x$. Therefore, Series (1) $= \sqrt{1+\tan^2 x} =$ secant x.

By integrating Series (1) with respect to x, we get

$$\int \sec x\cdot dx = x + \frac{1}{2\cdot 3}x^3 + \frac{5}{2\cdot 3\cdot 4\cdot 5}x^5 + \frac{61}{2\cdots 7}x^7 + \frac{1385}{2\cdots 9}x^9 + \ldots$$

It is worth noting the variety of expressions we have for $\int \sec x\cdot dx$. Here it is given in terms of the arc. By Table III, making $m = 1$, we have it in terms of the logarithm of the sine; by Series II, making $m = -1$, we have it in terms of the sine; by Art. 30, making $m = -1$, we have it in terms of the cosine; by Art. 33, making $m = 1$, we have it in terms of the secant; and by Art. 39, it is the solving element in the rectification of the logarithmic curve.

The integral of $\sec x\cdot dx$, by Art. 33, is $\log(\sec\cdot x + \tan\cdot x)$; let this $= z = x + \frac{1}{2\cdot 3}x^3 + \frac{5}{2\cdot 3\cdot 4\cdot 5}x^5 + \frac{61}{2\cdots 7}x^7 + \ldots$ as above. Now,

when this series is reversed, it has the same coefficients, with the signs alternately + and −;

$$x = z - \frac{1}{2\cdot 3} z^3 + \frac{5}{2\cdot 3\cdot 4\cdot 5} z^5 - \frac{61}{2\cdots 7} z^7 + \ldots$$

And $e^z - e^{-z} = 2 \tan\cdot x$; $e^z + e^{-z} = 2\cdot \sec\cdot x$.

23. The series commonly given in mathematical works, for expressing the tangent in terms of the arc, is derived from Euler's formulae for the sine and cosine. Let $i = \sqrt{-1}$, then these formulae are:

$$\sin x = \frac{1}{2i}(e^{xi} - e^{-xi}).$$

$$\cos x = \tfrac{1}{2}(e^{xi} + e^{-xi}).$$

And if the first be divided by the second, we get,

$$\tan x = \frac{1}{i}\left(1 - \frac{2}{1 + e^{2xi}}\right).$$

Now, if this last expression be expanded in a series, the result will be identical with series (4), Art. 20. Mathematicans, however, have not given the result in that form, evidently because of the difficulty with the law or coefficients; but they have given this equivalent form,

$$\tan x = 2^2(2^2 - 1)\frac{B_1 x}{2} + 2^4(2^4 - 1)\frac{B_2 x^3}{2\cdot 3\cdot 4} + 2^6(2^6 - 1)\frac{B_3 x^5}{2\cdot 3\cdot 4\cdot 5\cdot 6} + \ldots$$

where B_1, B_2, B_3, B_4 . . . are the celebrated Bernoullian Numbers.

The relation between these numbers and the numerators of coefficients is readily established by comparing the two series in which they respectively occur, namely:

$$N_a \cdot \frac{a+1}{2^{a+1}(2^{a+1} - 1)} = B_{\frac{a+1}{2}}.$$

24. Series (4), Art. 20, may be derived from another source, more elementary than the foregoing.

Take the simple formula in trigonometry,

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a\cdot \tan b}$$

and let $b = a, 2a, 3a \ldots ma$, in successive operations; then

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}; \quad \tan 3a = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a},$$

$$\tan 4a = \frac{4 \tan a - 4 \tan^3 a}{1 - 6 \tan^2 a + \tan^4 a}, \tan 5a = \frac{5 \tan a - 10 \tan^3 a + \tan^5 a}{1 - 10 \tan^2 a + 5 \tan^4 a}.$$

Here, the law of progression is this: for $\tan(ma)$, express $(1 + \tan a)^m$ in a series, and place the successive terms in the denominator and in the numerator alternately, and change the signs

so that they will be + and − alternately, in the denominator and also in the numerator. Now, in these expressions for multiple tangents, let us substitute arc $\frac{x}{n}$ for arc a. Let x represent any proportion of π, and let n be infinitely large. Then, $\tan\frac{x}{n}$, being infinitely small, does not differ from its arc, $\frac{x}{n}$. Hence, these multiple tangents may be written,

$$\tan\left(2\frac{x}{n}\right)=\frac{2\frac{x}{n}}{1-\frac{x^2}{n^2}};\quad \tan\left(3\frac{x}{n}\right)=\frac{3\frac{x}{n}-\frac{x^3}{n^3}}{1-3\frac{x^2}{n^2}};\quad \tan\left(4\frac{x}{n}\right)=\frac{4\frac{x}{n}-4\frac{x^3}{n^3}}{1-6\frac{x^2}{n^2}+\frac{x^4}{n^4}};$$

and, generally, $\tan\left(m\frac{x}{n}\right)=$

$$\frac{m\frac{x}{n}-\frac{m\cdot m-1\cdot m-2}{2\cdot 3}\cdot\frac{x^3}{n^3}+\frac{m\cdot m-1\cdot m-2\cdot m-3\cdot m-4}{2\cdot 3\cdot 4\cdot 5}\cdot\frac{x^5}{n^5}-\dots}{1-\frac{m\cdot m-1}{2}\cdot\frac{x^2}{n^2}+\frac{m\cdot m-1\cdot m-2\cdot m-3}{2\cdot 3\cdot 4}\cdot\frac{x^4}{n^4}-\frac{m\cdot m-1\dots m-5}{2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{x^6}{n^6}+\dots}.$$

Since this is true for every value of m, let m be considered infinitely large. Then $m-1$, $m-2$, $m-3$ . . . each equals m. Now, in each term of the series, m has the same power in the numerator of the coefficient as n has in the denominator of the argument; and, both being infinite, they unitize each other throughout, in both sides of the equation; and the series is, therefore, reduced to this,

$$\tan x=\frac{x-\frac{1}{2\cdot 3}x^3+\frac{1}{2\cdot 3\cdot 4\cdot 5}x^5-\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7+\dots}{1-\frac{1}{2}x^2+\frac{1}{2\cdot 3\cdot 4}x^4-\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6}x^6+\dots}.$$

But, since x represents any proportion of π, the numerator of this series expresses the sine, and the denominator the cosine, in terms of the arc; and if the one be divided by the other, the resulting series will be identical with Series (4), Art. 20. Therefore, this single operation on the theory of multiple tangents gives series for the sine, the cosine and the tangent, in terms of the arc.

On the Calculation of the Bernoullian Numbers

25. Definition. If the sum of the series, $1^p+2^p+3^p+4^p+\dots n^p$, be expressed in terms of the powers of n, the coefficients of n^1 in these expressions, for even values of p, will be the Bernoullian Numbers; $p=2$ giving B_1; $p=4$ giving B_2, etc. The sum of this series may be found in terms of n by the Theory of Differences,

as follows: Find the successive Orders of Differences in the series $a, b, c, d, e \ldots$, thus:

Series	1st Order d_1	2d Order d_2	3d Order d_3	4th Order d_4	5th Order d_5
a					
b	$b - a$				
c	$c - b$	$c - 2b + a$			
d	$d - c$	$d - 2c + b$	$d - 3c + 3b - a$		
e	$e - d$	$e - 2d + c$	$e - 3d + 3c - b$	$e - 4d + 6c - 4b + a$	
f	$f - e$	$f - 2e + d$	$f - 3e + 3d - c$	$f - 4e + 6d - 4c + b$	$f - 5e + 10d - 10c + 5b - a$

Here, the First Difference of the mth Order of Differences, d_m, is

$$\mp a \pm mb \mp \frac{m \cdot m - 1}{2} c \pm \frac{m \cdot m - 1 \cdot m - 2}{2 \cdot 3} d \mp \ldots$$

using the upper or lower sign according as m is odd or even. And the sum of the series $a + b + c + d + \ldots n$ is

$$na + \frac{n \cdot n - 1}{2} d_1 + \frac{n \cdot n - 1 \cdot n - 2}{2 \cdot 3} d_2 + \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3}{2 \cdot 3 \cdot 4} d_3 + \ldots,$$

where $d_1, d_2, d_3 \ldots$ are the First Differences in the respective Orders of Differences.

For $a + b + c + d + \ldots$, substitute the series, $1^p + 2^p + 3^p + 4^p + \ldots n^p = S$.

Let $p = 1$, then $S = \frac{1}{2}n^2 + \frac{1}{2}n$.

Let $p = 2$, then $S = n + \frac{n \cdot n - 1}{2} d_1 + \frac{n \cdot n - 1 \cdot n - 2}{2 \cdot 3} d_2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n$.

The coefficient of n^1, here, is $\frac{1}{6}$; therefore $B_1 = \frac{1}{6}$.

Let $p = 3$, then $S = n + \frac{n \cdot n - 1}{2} d_1 + \frac{n \cdot n - 1 \cdot n - 2}{2 \cdot 3} d_2 + \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3}{2 \cdot 3 \cdot 4} d_3 = \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} n^2$.

Let $p = 4$, then $S = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$.

The coefficient of n^1 in this case is $\frac{1}{30}$; therefore $B_2 = \frac{1}{30}$.

The successive Bernoullian Numbers may be found as above, but the labor increases very rapidly with the higher values of p.

Now, collect the foregoing results into one view, together with others derived from the same source, thus:

$p = 1$, $S = \frac{1}{2}n^2 + \frac{1}{2}n$.
$p = 2$, $S = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$.
$p = 3$, $S = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$.

$p = 4,\ S = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n.$
$p = 5,\ S = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2.$
$p = 6,\ S = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n.$
$p = 7,\ S = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2.$
$p = 8,\ S = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n.$
$p = 9,\ S = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{3}{20}n^2.$

In each of these values of S, it is evident that the

First term is $\frac{1}{p+1}N^{p+1}$.

Second term is $\frac{1}{2}N^p$.

Third term is $\frac{p}{2}\cdot\frac{1}{6}N^{p-1}, = \frac{p}{2}B_1N^{p-1}$.

Fourth term is $\frac{p\cdot p-1\cdot p-2}{2\cdot3\cdot4}\cdot\frac{1}{30}N^{p-3}, = \frac{p\cdot p-1\cdot p-2}{2\cdot3\cdot4}B_2\,N^{p-3}$.

Fifth term is $\frac{p\cdot p-1\cdot p-2\cdot p-3\cdot p-4}{2\cdot3\cdot4\cdot5\cdot6}\cdot\frac{1}{42}N^{p-5},$

$$= \frac{p\cdot p-1\cdot p-2\cdot p-3\cdot p-4}{2\cdot3\cdot4\cdot5\cdot6}B_3N^{p-5}.$$

Hence, we get the general equation, $1^p + 2^p + 3^p + 4^p + \ldots n^p$

$$= \frac{1}{p+1}N^{p+1} + \frac{1}{2}N^p + \frac{p}{2}B_1N^{p-1} - \frac{p\cdot p-1\cdot p-2}{2\cdot3\cdot4}B_2N^{p-3}$$
$$+ \frac{p\cdot p-1\ldots p-4}{2\cdot3\ldots6}B_3N^{p-5} - \frac{p\cdot p-1\ldots p-6}{2\cdot3\ldots8}B_4N^{p-7} + \ldots,$$

omitting the term N^0, which occurs when p is an odd number. Since this equation is true for all cases where p and n are positive integers, it is true when $n = 1$; that is, when the numerical side of the equation is reduced to the first term, namely, 1^p. Making $n = 1$, we get the equation

$$1 = \frac{1}{p+1} + \frac{1}{2} + \frac{p}{2}B_1 - \frac{p\cdot p-1\cdot p-2}{2\cdot3\cdot4}B_2$$
$$+ \frac{p\cdot p-1\cdot p-2\cdot p-3\cdot p-4}{2\cdot3\cdot4\cdot5\cdot6}B_3 - \ldots$$

This series always terminates with the term $B_{\frac{p}{2}}$. These Numbers may, therefore, be directly derived from it with greater facility than from the Theory of Differences; although that theory was necessary to establish the series.

Let $p = 2$, then $1 = \frac{1}{3} + \frac{1}{2} + B_1$. Therefore $B_1 = \frac{1}{6}$.
$p = 4$, then $1 = \frac{1}{5} + \frac{1}{2} + \frac{2}{6} - B_2$. $\therefore$ $B_2 = \frac{1}{30}$.
$p = 6$, then $1 = \frac{1}{7} + \frac{1}{2} + \frac{3}{6} - \frac{5}{30} + B_3$. $\therefore$ $B_3 = \frac{1}{42}$.
$p = 8$, then $1 = \frac{1}{9} + \frac{1}{2} + \frac{4}{6} - \frac{14}{30} + \frac{2}{9} - B_4$. $\therefore$ $B_4 = \frac{1}{30}$.
$p = 10$, then $1 = \frac{1}{11} + \frac{1}{2} + \frac{5}{6} - \frac{30}{30} + \frac{42}{42} - \frac{15}{30} + B_5$. $\therefore$ $B_5 = \frac{5}{66}$.

26. The first seventeen of these Numbers are given in the following table, together with the Numerators of Coefficients. They were calculated by this last series, and verified by an independent calculation of the Numerators by the law given in Art. 21 and 22.

NUMERATORS OF COEFFICIENTS for Odd and Even Powers of X, and the BERNOULLIAN NUMBERS Corresponding with the Odd Powers of X

X^a	N_a		$B_{\frac{a+1}{2}}$
X^0	1		
X^1	1	B_1	1/6
X^2	1		
X^3	2	B_2	1/30
X^4	5		
X^5	16	B_3	1/42
X^6	61		
X^7	272	B_4	1/30
X^8	1385		
X^9	7936	B_5	5/66
X^{10}	50521		
X^{11}	3 53792	B_6	691/2730
X^{12}	27 02765		
X^{13}	223 68256	B_7	7/6
X^{14}	1993 60981		
X^{15}	19037 57312	B_8	3617/510
X^{16}	1 93915 12145		
X^{17}	20 98653 42976	B_9	43867/798
X^{18}	240 48796 75441		
X^{19}	2908 88851 12832	B_{10}	1 74611/330
X^{20}	37037 11882 37525		
X^{21}	4 95149 80531 24096	B_{11}	8 54513/138
X^{22}	69 34887 43931 37901		
X^{23}	1015 42388 65068 52352	B_{12}	2363 64091/3730
X^{24}	15514 53416 35570 86905		
X^{25}	2 46921 48019 02079 83616	B_{13}	85 53103/6
X^{26}	40 87072 50929 31238 92361		
X^{27}	702 51601 60394 39598 87872	B_{14}	2 37494 61029/870
X^{28}	12522 59461 40362 98654 68285		
X^{29}	2 31191 84187 80959 78414 73536	B_{15}	861 58412 76005/14322
X^{30}	44 15438 93249 02310 45536 82821		
X^{31}	871 39627 57125 16929 61708 11392	B_{16}	770 93210 41217/510
X^{32}	17751 93915 79539 28943 66647 89665		
X^{33}	3 72940 77037 20529 57109 75096 25856	B_{17}	257 76878 58367/6

SECTION FOUR

Theorem for integrating the Powers of a Transcendental Function, in Terms of the Function considered as an Algebraic Quantity

27. In Series I and II, it will be observed, the integrals of $\sin^m x \cdot dx$ are expressed in terms of $\cos x$; and those for $\cos^m x \cdot dx$ are expressed in terms of $\sin x$. In my efforts to express these functions in their own terms, I found the following method to be of general application:

Let $y = \phi x$. Divide y by $\dfrac{dy}{dx}$ expressed in terms of y;

thus, $\dfrac{y}{\left(\dfrac{dy}{dx}\right)}$; raise y to the mth power and we have $\dfrac{y^m}{\left(\dfrac{dy}{dx}\right)}$.

Integrate this last expression, with respect to y, as an independent variable and as an algebraic function. The result will be the integral of $(\phi x)^m \cdot dx$, expressed in terms of the function itself, whether it be algebraic or transcendental.

28. Let $y = x^3$. $\dfrac{dy}{dx} = 3x^2 = 3y^{\frac{2}{3}}$. Then $\dfrac{y^m}{3y^{\frac{2}{3}}}$ integrated with respect to $y = \dfrac{y^{m+\frac{1}{3}}}{3(m + \frac{1}{3})}$. Therefore, the integral of $(x^3)^m = \dfrac{x^{3m+1}}{3m + 1}$.

29. Let $y = \sin x$. $\dfrac{dy}{dx} = \cos x = \sqrt{1 - y^2}$. Then $\dfrac{y^m}{\sqrt{1 - y^2}}$ being reduced to a series, and integrated with respect to y, gives:

$$\int \sin^m x \cdot dx = \frac{1}{m+1} S^{m+1} + \frac{1}{2(m+3)} S^{m+3} + \frac{3}{2\cdot 4(m+5)} S^{m+5}$$
$$+ \frac{3\cdot 5}{2\cdot 4\cdot 6(m+7)} S^{m+7} + \ldots, \text{ where } S \text{ represents } \sin x.$$

When m is negative, the constant C to be added is the full quadrant of the curvilinear for the particular value of m. Giving m particular values, we have

$$\int \sin^0 x \cdot dx = S + \frac{1}{2\cdot 3} S^3 + \frac{3}{2\cdot 4\cdot 5} S^5 + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 7} S^7 + \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 9} S^9 + \cdots = x.$$

$$\int \sin x \cdot dx = \frac{1}{2} S^2 + \frac{1}{2\cdot 4} S^4 + \frac{3}{2\cdot 4\cdot 6} S^6 + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 8} S^8 + \cdots = 1 - \cos x.$$

$$\int \sin^2 x \cdot dx = \frac{1}{3} S^3 + \frac{1}{2\cdot 5} S^5 + \frac{3}{2\cdot 4\cdot 7} S^7 + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 9} S^9 + \cdots$$
$$= \frac{x - \sin x \cdot \cos x}{2}.$$

30. Let $y = \cos x \cdot \dfrac{dy}{dx} = -\sin x = -\sqrt{1 - y^2}$. Then $\dfrac{y^m}{-\sqrt{1-y^2}}$, integrated with respect to y, gives

$$-\left(\frac{\cos^{m+1}x}{m+1} + \frac{\cos^{m+3}x}{2(m+3)} + \frac{3\cos^{m+5}x}{2\cdot 4(m+5)} + \frac{3\cdot 5\cos^{m+7}x}{2\cdot 4\cdot 6(m+7)} + \cdots\right).$$

This requires the constant C to be added, namely, the full quadrant of the curvilinear. But the full quadrant is the same for $\sin^m x$ as it is for $\cos^m x$; and for $\sin^m x$ the integral is

$$\frac{1^{m+1}}{m+1} + \frac{1^{m+3}}{2(m+3)} + \frac{3\cdot 1^{m+5}}{2\cdot 4\cdot(m+5)} + \frac{3\cdot 5\cdot 1^{m+7}}{2\cdot 4\cdot 6\cdot(m+7)} + \cdots$$

Now, adding this to the above series, we get

$$\int \cos^m x\cdot dx = \frac{(1-c^{m+1})}{m+1} + \frac{(1-c^{m+3})}{2(m+3)} + \frac{3(1-c^{m+5})}{2\cdot 4(m+5)} + \frac{3\cdot 5(1-c^{m+7})}{2\cdot 4\cdot 6(m+7)} + \cdots,$$

where c represents $\cos x$. This series eliminates the constant.

When m is an odd negative, these series will have an irrational term. In Art. 29, substitute for it $\log\sin x$; and in Art. 30, substitute for it $(1 - \log\cos x)$; retaining in each case the coefficient given by the series, excepting that the zero in the denominator is to be disregarded. The sum of these series may always be had from Tables I to IV, when m is a whole number, positive or negative.

31. Let $y = \tan x$. $\dfrac{dy}{dx} = 1 + t^2 = 1 + y^2$. And the integral of

$$\frac{y^m}{1+y^2} = \int \tan^m x = \frac{t^{m+1}}{m+1} - \frac{t^{m+3}}{m+3} + \frac{t^{m+5}}{m+5} - \cdots$$

This is identical with Series VII, and, therefore, the same conditions prevail as to the irrational term, constants and negative values.

32. Let $y = \sqrt{2x - x^2}$. Here x is an algebraic quantity; and this is the equation to the circle independently of trigonometrical functions; but its integral, nevertheless, will be the integral of the sine with respect to the versed sine of the arc.

$$\frac{dy}{dx} = \frac{1-x}{\sqrt{2x-x^2}} = \frac{1-x}{y} = \frac{\sqrt{1-y^2}}{y}; \text{ and } \frac{y^m}{\dfrac{\sqrt{1-y^2}}{y}} = \frac{y^{m+1}}{\sqrt{1-y^2}}.$$

Substituting S for y, the integral of this is,

$$\int \sin^m x\cdot d\cdot \operatorname{ver} x = \frac{S^{m+2}}{m+2} + \frac{S^{m+4}}{2(m+4)} + \frac{3\,S^{m+6}}{2\cdot 4\,(m+6)}$$
$$+ \frac{3\cdot 5\,S^{m+8}}{2\cdot 4\cdot 6(m+8)} + \cdots$$

Giving particular values to m,

$$m = 0;\ \frac{1}{2}S^2 + \frac{1}{2\cdot 4}S^4 + \frac{3}{2\cdot 4\cdot 6}S^6 + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 8}S^8 + \cdots = \text{versed sine of arc of which } S \text{ is the sine.}$$

$$m = 1;\ \frac{1}{3}S^3 + \frac{1}{2\cdot 5}S^5 + \frac{3}{2\cdot 4\cdot 7}S^7 + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 9}S^9 + \cdots = \frac{\text{arc} - \sin\cdot\cos}{2}.$$

$$m = 2;\ \frac{1}{4}S^4 + \frac{1}{2\cdot 6}S^6 + \frac{3}{2\cdot 4\cdot 8}S^8 + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 10}S^{10} + \cdots = \text{ver}^2 - \frac{1}{3}\text{ver}^3.$$

Taking the similar case, $y = \frac{b}{a}\sqrt{2ax - x^2}$, the equation to the ellipse; and omitting the constant $\frac{b}{a}$ for the present, we have,

$$\frac{dy}{dx} = \frac{a - x}{\sqrt{2ax - x^2}} = \frac{a - x}{y} = \frac{\sqrt{a^2 - y^2}}{y}.\quad \text{Then } \frac{b}{a}\int \frac{y^{m+1}}{\sqrt{a^2 - y^2}}$$

$$= \frac{b}{a^2}\left(\frac{y^{m+2}}{m + 2} + \frac{y^{m+4}}{2a^2(m + 4)} + \frac{3y^{m+6}}{2\cdot 4a^4(m + 6)} + \frac{3\cdot 5\ y^{m+8}}{2\cdot 4\cdot 6a^6(m + 8)} + \cdots\right).$$

Here, $y = \sqrt{2ax - x^2}$.
Making $m = 1$, we get,

$$\frac{b}{a^2}\left(\frac{1}{3}y^3 + \frac{1}{2\cdot 5a^2}y^5 + \frac{3}{2\cdot 4\cdot 7a^4}y^7 + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 9a^6}y^9 + \cdots\right).$$

This measures the area of the ellipse from the apex as origin. Therefore, if for any ordinate y, we make $\frac{y}{a} = \sin v$, then the area of the ellipse from the apex to that ordinate is $(a\cdot b)\cdot\dfrac{v - \sin v\cdot\cos v}{2}$.

If the full quadrant be considered, then $x = a = y$, and we get,

$$a\cdot b\left(\frac{1}{3} + \frac{1}{2\cdot 5} + \frac{3}{2\cdot 4\cdot 7} + \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 9} + \cdots\right).$$

The numerical series here is the same as in Art. 29, when $m = 2$ and $S = 1$. It, therefore, equals $\frac{\pi}{4}$ and $a\cdot b\cdot\frac{\pi}{4}$ is a quadrant of the ellipse whose semi-axes are a and b.

When x is greater than a, the series measures the complement of the second quadrant; y being a maximum when $x = a$.

33. Let $y = \sec x = \sqrt{1+t^2}\cdot$ $\dfrac{dy}{dx} = \dfrac{t+t^3}{\sqrt{1+t^2}} = y\cdot\sqrt{y^2-1}.$

Then $\dfrac{y^m}{y\cdot\sqrt{y^2-1}} = \dfrac{y^{m-1}}{\sqrt{y^2-1}}$; and the integral is

$$\frac{1}{m-1}y^{m-1} + \frac{1}{2(m-3)}y^{m-3} + \frac{3}{2\cdot 4(m-5)}y^{m-5} + \frac{3\cdot 5}{2\cdot 4\cdot 6(m-7)}y^{m-7} + \cdots$$

Giving particular values to m,
$m = 1$; then $\int \sec x\cdot dx =$

$$\log y - \frac{1}{2\cdot 2y^2} - \frac{3}{2\cdot 4\cdot 4y^4} - \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 6y^6} - \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 8y^8} - \cdots$$

Here, when $x = 0$, the area $= 0$, and $y = 1$; therefore $\log y = 0$. Hence, the constant, to be added, is the negative part of the series taken positively. Therefore, $C = +\log 2$. But, for any value of x, the negative part of the series is $\log[1+(\sqrt{1+t^2}-t)^2] = \log 2(1+t^2 - t\cdot\sqrt{1+t^2})$.

Therefore,
$\int \sec x\cdot dx = \log\sqrt{1+t^2} - \log(1+t^2-t\sqrt{1+t^2}) = \log(\sqrt{1+t^2}+t)\cdot$
Hence, the constant is eliminated.

Let $m = 3$, then

$$\int \sec^3 x\cdot dx = \frac{1}{2}y^2 + \frac{1}{2}\log y - \frac{3}{2\cdot 4\cdot 2y^2} - \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 4y^4} - \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 6y^6} - \cdots$$

The constant C, to be added, is $\frac{1}{2}\log 2 - \frac{1}{4}$.
But, for odd values of m,

$$\int \sec^{m+2}x\cdot dx = \frac{m}{m+1}\int \sec^m x\cdot dx + \frac{1}{m+1}\cdot\tan x\cdot\sec^m x.$$

Hence,
$\int \sec^3 x\cdot dx = \frac{1}{2}\log\sqrt{1+t^2} + \frac{1}{2}t\cdot\sqrt{1+t^2} - \frac{1}{2}\log(1+t^2-t\cdot\sqrt{1+t^2})$
$= \frac{1}{2}\log(\sqrt{1+t^2}+t) + \frac{1}{2}t\cdot\sqrt{1+t^2}.$

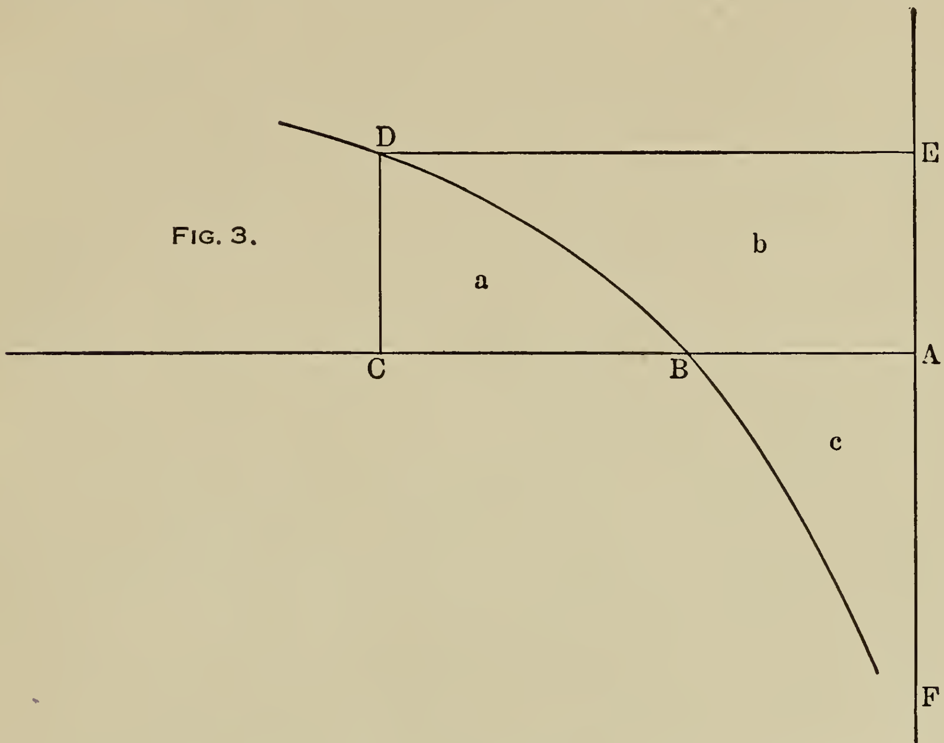
The sum of the series for other values of m may be had from Tables II and III and Series II, observing that $\sec^m x = \dfrac{1}{\cos^m x}\cdot$

34. On the integration of the functions $x = \log y$, and $y = \log x$. In this case, the two curves are the same, but the areas of their curvilinears are different; the first curve being concave, and the second convex, to the axis.

Fig. 3 represents the two curves, with A as origin of coördinates in each. In the equation $x = \log y$, $AE = x$, positive; $AF = x$, negative; $ED = y$. In the equation $y = \log x$, $AC = x$; $CD = y$; $AB = 1$. y is therefore negative when x is less than 1.

The two curves coincide throughout; and (omitting c, which represents the area for negative values of the functions) the sum of

FIG. 3.



the areas a and b is always equal to the parallelogram whose sides are x and y; that is $= x \cdot \log x$, or its equal, $y \cdot \log y$. And, in like manner, if the two curves $y = \log^m x$ and $x = \log^m y$, be traced, the sum of the corresponding areas a and b will be the parallelogram $x \cdot \log^m x$, or its equal $y \cdot \log^m y$.

35. Let $y = \log x$. $\dfrac{dy}{dx} = \dfrac{1}{x} \cdot \quad \dfrac{y}{\frac{dy}{dx}} = y \cdot x$. But $x = e^y$.

Therefore $\int \log^m x \cdot dx = \int y^m \cdot e^y$

$$= \frac{1}{m+1} y^{m+1} + \frac{1}{m+2} y^{m+2} + \frac{1}{2(m+3)} y^{m+3} + \frac{1}{2 \cdot 3(m+4)} y^{m+4}$$
$$+ \frac{1}{2 \cdot 3 \cdot 4(m+5)} y^{m+5} + \cdots$$

This series measures the curvilinear area from the point B, towards the left when x is greater than 1, and towards the right when x is less than 1.

The sum of this series for particular values of m, is

$\int \log x \cdot dx = x \cdot \log x - x + 1.$

$\int \log^2 x \cdot dx = x \cdot \log^2 x - 2x \cdot \log x + 2x - 2.$

$\int \log^3 x \cdot dx = x \cdot \log^3 x - 3x \cdot \log^2 x + 6x \cdot \log x - 6x + 6.$

$\int \log^4 x \cdot dx = x \cdot \log^4 x - 4x \cdot \log^3 x + 12x \cdot \log^2 x - 24x \cdot \log x + 24x - 24.$

And, generally, making $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots m = q$,

$$\int \log^m x \cdot dx = \pm q(x \cdot \log x - x + 1) \mp \frac{q}{2} x \cdot \log^2 x \pm \frac{q}{2 \cdot 3} x \cdot \log^3 x$$

$$\mp \frac{q}{2 \cdot 3 \cdot 4} x \cdot \log^4 x \pm \cdots \text{to term } \frac{q}{q} x \cdot \log^m x;$$ using the upper or lower sign according as m is odd or even.

36. In the equation $x = \log^m y$, we get $\sqrt[m]{x} = \log y$. (1)

Therefore, $y = e^{\sqrt[m]{x}}$; and $\dfrac{dy}{dx} = \dfrac{1}{m(x)^{\frac{m-1}{m}}} \cdot e^{\sqrt[m]{x}}$.

Hence, $\dfrac{y}{\frac{dy}{dx}} = m \cdot x^{\frac{m-1}{m}} = m(\sqrt[m]{x})^{m-1} = m \cdot \log^{m-1} y$ (1)

The required integral, therefore, is $m \int \log^{m-1} y$. This measures the area b, (Fig. 3), from the point A, when y is greater than 1; and the area c, when y is less than 1. Giving particular values to m, and supplying the constants, we get the area for

$m = 1$. $y - 1$.
$m = 2$. $2y \cdot \log y - 2y + 2$.
$m = 3$. $3y \cdot \log^2 y - 6y \cdot \log y + 6y - 6$.

And, making q = the product of the consecutive numbers from 1 to m, the area for $\int \log^m y \cdot dy = \mp q(y \cdot \log y - y + 1) \pm \frac{q}{2} y \cdot \log^2 y$

$$\mp \frac{q}{2 \cdot 3} y \cdot \log^3 y \pm \cdots \text{to the term } m \cdot y \cdot \log^{m-1} y,$$ using the upper or lower sign according as m is odd or even.

If the integral given here for any particular value of m be added to that given in Art. 35, the sum will be the area of the parallelogram $x \cdot \log^m x$, or $y \cdot \log^m y$.

Wherever logarithms are mentioned throughout this work, the natural logarithms are intended; that is, the system whose base is $e = 2 \cdot 718281828 \cdots$ But the logarithmic integrals given in this and the preceding article will equally apply to any other system of logarithms, if the complete expressions for these integrals be multiplied by the mth power of the modulus of that system, thus:

Let a be the base of any system of logarithms and M the modulus of that system; then the integrals given in Art. 35 become

$\int \log_a x \cdot dx = M(x \cdot \log_e x - x + 1) = x \cdot \log_a x - Mx + M$.

$\int \log_a{}^2 x \cdot dx = M^2(x \cdot \log_e{}^2 x - 2x \cdot \log_e x + 2x - 2) = x \cdot \log_a{}^2 x - 2M \cdot x \cdot \log_a \cdot x + 2M^2 x - 2M^2$, etc.

And in Art. 36, the integrals become $M \cdot y - M$.

$$M^2(2y \cdot \log_e y - 2y + 2) = 2M \cdot y \cdot \log_a y - 2M^2 y + 2M^2, \text{ etc.}$$

It may be observed that the constant quantity in each of the integrals given in articles 35 and 36 represents the whole area (c) between the curve and its asymptote $A\,F$; and it is always the product of the consecutive numbers from 1 to m, $= 1\cdot 2\cdot 3\cdot 4\cdots m$, $= q$ (multiplied by the mth power of the modulus of the system).

Hence, between the limits of $x = 0$ and 1, we have

$\int_0^1 \log^m\cdot x\cdot dx = 1\cdot 2\cdot 3\cdot 4\cdots m.$

Rectification of Curves

37. In the foregoing we have applied the theorem under consideration only to the quadrature of curves; but it may also be applied with advantage to the rectification of curves. It is well known that, from the equation $y = \phi x$, a secondary equation may be derived, namely: $y = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$; and the area of the curvilinear in this second equation will be equal to the length of the curve in the first equation.

To determine the length of the curve, $y = x^2$, we may integrate by the theorem, the equation $y = \sqrt{1 + 4x^2}$.

$$\text{Then, } \frac{dy}{dx} = \frac{4x}{\sqrt{1 + 4x^2}}; \text{ and } \frac{y}{\frac{dy}{dx}} = \frac{y^2}{4x} = \frac{y^2}{2\sqrt{y^2 - 1}}.$$

The integral of this is

$$\frac{1}{4}y^2 + \frac{1}{4}\log y - \frac{3}{4\cdot 4\cdot 2\cdot y^2} - \frac{3\cdot 5}{4\cdot 4\cdot 6\cdot 4\cdot y^4} - \frac{3\cdot 5\cdot 7}{4\cdot 4\cdot 6\cdot 8\cdot 6y^6} - \cdots \tag{1}$$

The sum of this series may be had by comparing it with the series given in Art. 33, for the integral of $\sec^3 x\cdot dx$, namely:

$$\frac{1}{2}y^2 + \frac{1}{2}\log y - \frac{3}{2\cdot 4\cdot 2y^2} - \frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 4\cdot y^4} - \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 6\cdot y^6} - \cdots \tag{2}$$

If $2x$ in Series (1) be considered equal to t in Series (2), then the sum of Series (1) will be half the sum of Series (2). Hence, we get the sum of Series (1) =

$$= \frac{1}{8}\log\frac{\sqrt{1 + 4x^2} + 2x}{\sqrt{1 + 4x^2} - 2x} + \frac{1}{2}x\cdot\sqrt{1 + 4x^2}, \text{ or, by reducing,}$$

$$= \frac{1}{4}\log(\sqrt{1 + 4x^2} + 2x) + \frac{1}{2}x\cdot\sqrt{1 + 4x^2}.$$

Now, when $x = 0$, the curve is zero, and this quantity is also zero. The constant C is therefore zero. And the length of the curve in the equation $y = x^2$ is

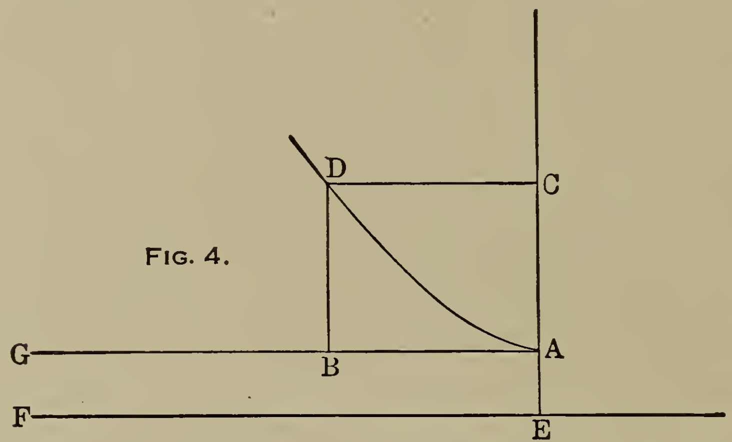
$$\tfrac{1}{4}\log(\sqrt{1 + 4x^2} + 2x) + \tfrac{1}{2}x\cdot\sqrt{1 + 4x^2},$$

and by introducing the constant A, the primary equation will be $y = Ax^2$. Then the length of the curve is

$$\frac{1}{4A}\cdot\log\cdot(\sqrt{1+4A^2x^2}+2Ax)+\tfrac{1}{2}x\cdot\sqrt{1+4A^2x^2}.$$

38. The curve in the preceding case is the same as the curve of the parabola whose parameter is 1. The equation to that particular parabola is $y = \sqrt{x}$. The two curvilinears form the parallelogram whose sides are x and y; and the two curves coincide throughout.

FIG. 4.



In Fig. 4, $AB = x$, $BD = y$, in the equation $y = x^2$.
$AC = x$, $CD = y$, in the equation $y = \sqrt{4ax}$; $AE = a = \frac{1}{4}$.
Since $BD = AB^2 \therefore DC = \sqrt{AC} = \sqrt{4ax}$.

That a similar relation between the two curves prevails for other values of the parameter, may be shown thus: $y = \sqrt{4ax}$, $\therefore x = \dfrac{y^2}{4a}\cdot$ Applying this form to the equation $y = x^2$, we get $y = \dfrac{x^2}{4a}$; which differs from Art. 37 only by the constant $\dfrac{1}{4a}\cdot$ From this we get the secondary equation $y = \sqrt{1+\dfrac{1}{4a^2}x^2}$; the integral of which (by repeating the operation in Art. 37) is

$$2a\left[\frac{1}{2}\log.\left(\sqrt{1+\frac{x^2}{4a^2}}+\frac{x}{2a}\right)+\frac{x}{4a}\cdot\sqrt{1+\frac{x^2}{4a^2}}\right. = a\cdot\log\left(\sqrt{1+\frac{x^2}{4a^2}}+\frac{x}{2a}\right)$$
$$+\frac{1}{2}x\cdot\sqrt{1+\frac{x^2}{4a^2}}\cdot$$

Here, AG is the axis of x. This measures the length of the curve in any common parabola, where a represents the distance from the directrix to the principal vertex.

From the foregoing may be derived the law that Projectiles move in parabolic curves. Let the moving body receive its momentum at A, in the direction AG; and let $\frac{x}{2\sqrt{a}}$ represent the time, the duration of the motion. The force of gravity deflects the body from the line AG towards the point D. The deflection is, therefore, BD. But $BD = \frac{x^2}{4a}\cdot$ Hence, the deflection is in proportion to the square of the time.

39. To determine the length of the logarithmic curve. In Fig. 3, the curve cuts the axis of x at the point B. Let this be considered the point of origin of coördinates; and the equation to the curve will be $y = \log(x + 1)$. But the equation $x = \log\cdot y$ gives the same curve, when EF is the axis of x, and A is the point of origin.

Therefore, in the equation $x = \log\cdot y$, $y = e^x$, and $\frac{dy}{dx} = e^x$.

Then the secondary equation, $y = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ becomes $y = \sqrt{1 + e^{2x}}$.

Here, $\frac{dy}{dx} = \frac{e^{2x}}{\sqrt{1 + e^{2x}}}$; and $\frac{y}{\frac{dy}{dx}} = \frac{y^2}{e^{2x}} = \frac{y^2}{y^2 - 1} = 1 + \frac{1}{y^2} + \frac{1}{y^4} + \frac{1}{y^6} + \cdots$

And the integral of this is

$$y - \frac{1}{y} - \frac{1}{3y^3} - \frac{1}{5y^5} - \frac{1}{7y^7} - \cdots \qquad (1)$$

Now, by Series II, by making $m = -1$, we have

$$\int \frac{1}{\cos x} = S + \frac{1}{3}S^3 + \frac{1}{5}S^5 + \frac{1}{7}S^7 + \cdots = \frac{1}{2}\log\frac{1 + \sin x}{1 - \sin x}\cdot \qquad (2)$$

Here, if we make $\sin x = \frac{1}{y}$, which is always possible, since $y(=\sqrt{1 + e^{2x}})$ is always greater than 1; then Series (2) will be equal to the negative part of Series (1).

Hence, the sum of Series (1) is,

$$y - \frac{1}{2}\log\frac{y + 1}{y - 1} + C.$$

Substitute for y, the differential of the curve in the equation $y = \log(x + 1)$, namely: $\sqrt{1 + e^{2y}} = \sqrt{1 + (1 + x)^2}$, and we get

$$\sqrt{1 + (1 + x)^2} - \frac{1}{2}\log\frac{\sqrt{1 + (1 + x)^2} + 1}{\sqrt{1 + (1 + x)^2} - 1} + C.$$

Now, when $x = 0$, the curve is zero. Therefore,

$$C = -\sqrt{2} + \frac{1}{2}\log\frac{\sqrt{2+1}+1}{\sqrt{2+1}-1} = -\sqrt{2} + \log(\sqrt{2}+1).$$

The length of the logarithmic curve is, therefore,

$$\sqrt{1+(1+x)^2} - \frac{1}{2}\log\frac{\sqrt{1+(1+x)^2}+1}{\sqrt{1+(1+x)^2}-1} - \sqrt{2} + \log(\sqrt{2}+1)$$

$$= \sqrt{1+(1+x)^2} - \log.\frac{\sqrt{1+(1+x)^2}+1}{1+x} - \sqrt{2} + \log.(\sqrt{2}+1).$$

This measures the length of the logarithmic curve from the point B (Fig. 3), towards the left for any positive value of x; and from the point B towards the right, when x has any negative value from 0 to -1. And, in the equation $x = \log y$, the length of the curve is

$$\sqrt{1+y^2} - \log\frac{\sqrt{1+y^2}+1}{y} - \sqrt{2} + \log(\sqrt{2}+1),$$

where y is the ordinate $D\,E$, and x the abscissa $A\,E$.

40. To determine the length of the curve for any system of logarithms whose base is a and modulus M. The primary equation is $y = \log_a(x+1), = M\cdot\log_e(x+1)$.

$$\frac{y}{M} = \log_e(x+1). \quad \therefore\ (x+1) = e^{\frac{y}{M}}. \quad \frac{dx}{dy} = \frac{1}{M}\cdot e^{\frac{y}{M}}.$$

The secondary equation is, therefore, $y = \sqrt{1 + \frac{1}{M^2}e^{\frac{2y}{M}}}$.

From this we get $\dfrac{y^2}{\frac{1}{M^3}e^{\frac{2y}{M}},} = \dfrac{M\cdot y^2}{y^2-1}$.

And the integral of this, as in Art. 39, is

$$M\cdot\left(y - \frac{1}{2}\log_e\frac{y+1}{y-1}\right) + C.$$

Substitute for y, its value in terms of $(x+1)$, and we have

$$M\cdot\left(\sqrt{1+\frac{1}{M^2}(1+x)^2} - \frac{1}{2}\log_e\frac{\sqrt{1+\frac{1}{M^2}(1+x)^2}+1}{\sqrt{1+\frac{1}{M^2}(1+x)^2}-1}\right) + C.$$

When $x = 0$, the curve is zero; hence,

$$C = -M\left(\sqrt{1+\frac{1}{M^2}} - \frac{1}{2}\log_e\frac{\sqrt{1+\frac{1}{M^2}}+1}{\sqrt{1+\frac{1}{M^2}}-1}\right)$$

The length of the curve is, therefore,

$$M\left[\sqrt{1+\frac{1}{M^2}(1+x)^2} - \frac{1}{2}\log_e \frac{\sqrt{1+\frac{1}{M^2}(1+x)^2}+1}{\sqrt{1+\frac{1}{M^2}(1+x)^2}-1} - \sqrt{1+\frac{1}{M^2}} + \right.$$

$$\left.\log_e M\left(\sqrt{1+\frac{1}{M^2}}+1\right)\right].$$

And, in terms of $\log_a$ this expression becomes

$$\sqrt{M^2+(1+x)^2} - \frac{1}{2}\log_a \frac{\sqrt{M^2+(1+x)^2}+M}{\sqrt{M^2+(1+x)^2}-M} - \sqrt{M^2+1}$$

$$+ \log_a(\sqrt{M^2+1}+M) = \sqrt{M^2+(1+x)^2}$$

$$- \log_a \frac{\sqrt{M^2+(1+x)^2}+M}{1+x} - \sqrt{M^2+1} + \log_a(\sqrt{M^2+1}+M).$$

SECTION FIVE

On the Integration of the Logarithms of Binomials, and other Complex Quantities.

41. In considering the following logarithmic integrals as the measure of curvilinear areas, it is necessary to observe from what point in the axis the measure is reckoned. In Fig. 3, Art. 34, the line AC is the axis of x, and the curve cuts the axis at the point B. The form of the curve varies with the nature of the function; but, whatever may be the form of the function, the curve will cross the axis at that point where x (from origin A) has such a value as to make the whole function equal 1, because the logarithm of $1 = 0$. Hence, these integrals measure the area (a) above the axis, when the value of x makes function x greater than 1; and they measure the area (c) below the axis, when the value of x makes function x less than 1, but not negative.

42. The integral of $\log(1 + x) = (1 + x)\cdot\log(1 + x) - x$.

The integral of $\log(1 - x) = -(1 - x)\cdot\log(1 - x) - x$.

When $x = 1$, this last integral has the term $0\cdot\log 0$; apparently irrational; but its value can be defined, thus: $(1 - x)\cdot\log(1 - x) = \log(1 - x)^{(1-x)} = \log(1 - x)^0 = \log 1, = 0$; hence, when $x = 1$, the whole value of this last integral $= -1$. That is, the logarithm of an infinite number, multiplied by zero, = nothing.

To find the integral of $\log(1 + x^2)$, assume $\int\log(1 + x^2) = x\cdot\log(1 + x^2) + z$, and differentiate, thus:

$$\log(1 + x^2) = \log(1 + x^2) + \frac{2x^2}{1 + x^2} + d\cdot z;$$

hence,

$$z = -\int\frac{2x^2}{1 + x^2} = -2x + 2v;$$

therefore,

$$\int\cdot\log(1 + x^2) = x\cdot\log(1 + x^2) + 2v - 2x,$$

when v is the arc of which x is the tangent.

To find the integral of $\log(1 - x^2)$, assume

$$\int\log(1 - x^2) = x\cdot\log(1 - x^2) + z,$$

and differentiate, thus,

$$\log(1 - x^2) = \log(1 - x^2) - \frac{2x^2}{1 - x^2} + d\cdot z;$$

hence,

$$z = \int \frac{2x^2}{1-x^2} = \log \frac{1+x}{1-x} - 2x;$$

therefore,

$$\int \log (1-x^2) = x \cdot \log (1-x^2) + \log \frac{1+x}{1-x} - 2x.$$

This integral may also be had as follows:
$1-x^2 = (1+x)\cdot(1-x)$ $\therefore \int \log(1-x^2) = \int \log (1+x)$ $+ \int \log(1-x) = (1+x)\cdot\log(1+x) - (1-x)\log(1-x) - 2x.$

In both forms of this last integral, there are quantities apparently irrational when $x = 1$. In the second form, it has been shown that $(1-x)\cdot\log(1-x) = 0$, when $x = 1$, and the whole integral is, therefore, $2 \cdot \log 2 - 2$.

In the first form $\log \frac{1+x}{1-x}$ becomes $\log \frac{(1+x)^2}{1-x^2} = 2 \cdot \log (1+x)$ $- \log(1-x^2)$. Hence, the whole integral becomes,

$$x \cdot \log (1-x^2) + 2 \log (1+x) - \log (1-x^2) - 2x$$

and when $x = 1$ this equals $2 \cdot \log (1+x) - 2x = 2 \cdot \log 2 - 2$, as above.

43. By introducing constants into these functions, and pursuing the same methods, we get

$$\int \log (a+x) = (a+x) \log (a+x) - x - a \cdot \log a.$$

$$\int \log (a-x) = -(a-x) \log (a-x) - x + a \cdot \log a.$$

$$\int \log (a+x^2) = x \cdot \log (a+x^2) - 2x + 2\sqrt{a} \cdot v,$$

when v is the arc of which $\frac{x}{\sqrt{a}}$ is the tangent.

$$\int \log (a-x^2) = x \cdot \log (a-x^2) + \sqrt{a} \cdot \log \frac{\sqrt{a}+x}{\sqrt{a}-x} - 2x$$

$$= 2\sqrt{a} \log(\sqrt{a}+x) - 2x, \text{ when } x = \sqrt{a}.$$

$$\int \log (a^2+x^2) = x \cdot \log (a^2+x^2) - 2x + 2a \cdot v,$$

when v is the arc of which $\frac{x}{a}$ is the tangent.

$$\int \log (a^2-x^2) = x \cdot \log (a^2-x^2) + a \cdot \log \frac{a+x}{a-x} - 2x$$

$= 2a \cdot \log (a+x) - 2x$, when $x = a$.

$$\int \log (a^2+b^2x^2) = x \cdot \log (a^2+b^2x^2) - 2x + \frac{2a}{b} \cdot v,$$

when v is the arc of which $\frac{bx}{a}$ is the tangent.

$$\int \log (a^2-b^2x^2) = x \cdot \log (a^2-b^2x^2) - 2x + \frac{a}{b} \cdot \log \frac{a+bx}{a-bx}$$

$= 2a \cdot \log (a+bx) - 2x$, when $bx = a$.

It will be observed that when x has such a value as to make the whole function = 0, the integral contains two logarithmic terms, which, when taken together, can always be valued; but not when taken separately, because each involves the logarithm of an infinite number. Therefore, the whole area between the curve and its asymptote can be determined for each of these functions.

The asymptote to the curve in each of these functions is the perpendicular dropped from that point of the axis where x has such a value as to make the whole function equal nothing.

44. Theorem. Let w equal any function of x, simple or complex, algebraic or transcendental, then the integral, with respect to x, of the product of $\frac{dw}{dx}$ and $\log^m w$ will equal the integral of $\log^m(w)$ with respect to w treated as a simple algebraic variable. That is,

$$\int \frac{dw}{dx} \cdot \log^m w = \int \log^m(w).$$

The forms of this last integral, for particular values of m, may be had from Art. 35; for example,

$$\text{let } m = 1, \text{ and } w = x + \sin x. \quad \frac{dw}{dx} = 1 + \cos x;$$

then

$$\int (1 + \cos x) \cdot \log (x + \sin x) = (x + \sin x) \cdot \log (x + \sin x) - (x + \sin x) + 1.$$

Let

$$w = 1 + x^2. \quad \frac{dw}{dx} = 2x,$$

then

$$\int 2x \cdot \log (1 + x^2) = (1 + x^2) \cdot \log (1 + x^2) - x^2.$$

Let

$$w = \tan \cdot x. \quad \frac{dw}{dx} = 1 + \tan^2 x;$$

then

$$\int (1 + \tan^2 x) \log \tan x = \tan \cdot x (\log \tan x) - \tan x + 1.$$

Let

$$w = \log x. \quad \frac{dw}{dx} = \frac{1}{x},$$

then

$$\int \left(\frac{1}{x} \cdot \log \text{ of } \log x\right) = \log \cdot x (\log \text{ of } \log x) - \log x + 1.$$

Let

$$m = 2, \text{ and } w = 1 + x^3,$$

then

$\int 3x^2 \cdot \log^2(1 + x^3) = (1 + x^3) \cdot \log^2(1 + x^3) - (2 + 2x^3) \cdot \log(1 + x^3)$
$+ 2x^3.$

45. The integrals of certain classes of algebraic fractions may be readily had by means of the foregoing theorem.

Let

$$w = (1 + x^m). \quad \frac{dw}{dx} = m \cdot x^{m-1};$$

and, by the theorem,

$$\int m \cdot x^{m-1} \cdot \log(1 + x^m) = (1 + x^m) \cdot \log(1 + x^m) - x^m.$$

Assume this equals $x^m \cdot \log(1 + x^m) + z$, and differentiate.

$$m \cdot x^{m-1} \cdot \log(1 + x^m) = m \cdot x^{m-1} \cdot \log(1 + x^m) + \frac{m \cdot x^{2m-1}}{1 + x^m} + d \cdot z.$$

Hence,

$$m \int \frac{x^{2m-1}}{1 + x^m} = -z, = x^m - \log(1 + x^m).$$

Therefore

$$\int \frac{x^{2m-1}}{1 + x^m} = \frac{1}{m}\left(x^m - \log(1 + x^m)\right).$$

And making $w = (1 - x^m)$, by repeating the process, we have

$$\int \frac{x^{2m-1}}{1 - x^m} = -\frac{1}{m}\left(x^m + \log(1 - x^m)\right)$$

Giving particular values to m, we get,

$$m = 1. \quad \int \frac{x}{1 + x} = x - \log(1 + x).$$

$$\int \frac{x}{1 - x} = -\left(x + \log(1 - x)\right)$$

$$m = -1. \quad \int \frac{x^{-3}}{1 + x^{-1}} = \int \frac{1}{x^3 + x^2} = -\left(\frac{1}{x} - \log\left(1 + \frac{1}{x}\right)\right)$$

$$\int \frac{1}{x^3 - x^2} = \frac{1}{x} + \log\left(1 - \frac{1}{x}\right).$$

$$m = \frac{1}{2}. \quad \int \frac{x^0}{1 + x^{\frac{1}{2}}} = \int \frac{1}{1 + \sqrt{x}} = 2\left(\sqrt{x} - \log(1 + \sqrt{x})\right).$$

$$\int \frac{1}{1 - \sqrt{x}} = -2\left(\sqrt{x} + \log(1 - \sqrt{x})\right).$$

$$m = -\frac{1}{2}. \quad \int \frac{x^{-2}}{1 + \frac{1}{\sqrt{x}}} = \int \frac{1}{x^2 + x^{\frac{3}{2}}} = -2\left(\frac{1}{\sqrt{x}} - \log\left(1 + \frac{1}{\sqrt{x}}\right)\right).$$

$$\int \frac{1}{x^2 - x^{\frac{3}{2}}} = 2\left(\frac{1}{\sqrt{x}} + \log\left(1 - \frac{1}{\sqrt{x}}\right)\right).$$

$$m = \pm\sqrt{5} \quad \int \frac{x^{2\sqrt{5}-1}}{1 + x^{\sqrt{5}}} = \frac{1}{\sqrt{5}}\left(x^{\sqrt{5}} - \log\left(1 + x^{\sqrt{5}}\right)\right).$$

$$\int \frac{x^{2\sqrt{5}-1}}{1 - x^{\sqrt{5}}} = -\frac{1}{\sqrt{5}}\left(x^{\sqrt{5}} + \log\left(1 - x^{\sqrt{5}}\right)\right).$$

$$\int \frac{1}{x^{2\sqrt{5}+1} + x^{\sqrt{5}+1}} = -\frac{1}{\sqrt{5}}\left(\frac{1}{x^{\sqrt{5}}} - \log\left(1 + \frac{1}{x^{\sqrt{5}}}\right)\right)$$

$$\int \frac{1}{x^{2\sqrt{5}+1} - x^{\sqrt{5}+1}} = \frac{1}{\sqrt{5}}\left(\frac{1}{x^{\sqrt{5}}} + \log\left(1 - \frac{1}{x^{\sqrt{5}}}\right)\right).$$

46. The general formulae given in the preceding article may be varied as follows:

$$\int \frac{x^{m-1}}{1 + x^m} = \frac{1}{m}\cdot\log(1 + x^m).$$

$$\int \frac{x^{m-1}}{1 - x^m} = -\frac{1}{m}\log(1 - x^m).$$

$$\int \frac{x^{3m-1}}{1 + x^m} = \frac{1}{2m}x^{2m} - \frac{1}{m}\left(x^m - \log(1 + x^m)\right)$$

$$\int \frac{x^{3m-1}}{1 - x^m} = -\frac{1}{2m}x^{2m} - \frac{1}{m}\left(x^m + \log(1 - x^m)\right)$$

$$\int \frac{x^{4m-1}}{1 + x^m} = \frac{1}{m}\left(x^m - \log(1 + x^m)\right) - \frac{1}{2m}x^{2m} + \frac{1}{3m}x^{3m}.$$

$$\int \frac{x^{4m-1}}{1 - x^m} = -\frac{1}{m}\left(x^m + \log(1 - x^m)\right) - \frac{1}{2m}x^{2m} - \frac{1}{3m}x^{3m}.$$

47. Let

$$w = a + bx^m + cx^n \cdot \quad \frac{dw}{dx} = mbx^{m-1} + ncx^{n-1}.$$

By the theorem,

$$\int (mbx^{m-1} + ncx^{n-1})\cdot\log(a + bx^m + cx^n)$$
$$= (a + bx^m + cx^n)\cdot\log(a + bx^m + cx^n) - (a + bx^m + cx^n) + 1.$$

Assume this integral equals

$$(bx^m + cx^n)\cdot\log(a + bx^m + cx^n) + z$$

and differentiate.

Then

$$\int \frac{mb^2x^{2m-1} + (m + n)\,bcx^{m+n-1} + nc^2x^{2n-1}}{a + bx^m + cx^n} = -z$$

$$= bx^m + cx^n - a\cdot\log(a + bx^m + cx^n) + a\cdot\log a.$$

The constants a, b, c, m, n may be positive or negative, whole numbers or fractions.

To give an example of this last integral:
Let

$$a = 1,\ b = -2,\ c = 3,\ m = -4,\ n = 5;$$

then

$$\int \frac{45x^9 - \dfrac{16}{x^9} - 6}{1 - \dfrac{2}{x^4} + 3x^5} = 3x^5 - \frac{2}{x^4} - \log\left(1 - \frac{2}{x^4} + 3x^5\right).$$

The foregoing methods may be applied to any function of two or more variables, however complex it may be, provided the function contains at least one constant not affected by the variable.

48. The theorem under consideration may also be used to find the integrals of certain classes of fractions containing circular functions.

Let v represent any circular arc in the first quadrant, and $s = \sin v$, and $t = \tan v$.

Take the function $w = (a + b \sin^m v)$,

$$\frac{dw}{dv} = bm \cdot s^{m-1} \cdot \sqrt{1 - s^2}.$$

Then, by the theorem,

$\int \sqrt{1 - s^2} \cdot bm \cdot s^{m-1} \cdot \log(a + bs^m) = (a + bs^m) \log(a + bs^m) - (a + bs^m) + 1.$

Assume this equals $bs^m \cdot \log(a + bs^m) + z$, and differentiate; then

$$\int \frac{\sqrt{1 - s^2} \cdot b^2 s^{2m-1}}{a + bs^m} = -z = bs^m - a \cdot \log(a + bs^m) + a \cdot \log a.$$

Take the function $w = a + b \tan^m v$.

$$\frac{dw}{dv} = bm(t^{m-1} + t^{m+1}).$$

Then, by the theorem,

$\int bm(t^{m-1} + t^{m+1}) \cdot \log(a + bt^m) = (a + bt^m) \cdot \log(a + bt^m) - (a + bt^m) + 1.$

Assume this equals $bt^m \cdot \log(a + bt^m) + z$, and differentiate.
Then

$$\int \frac{b^2 \cdot m(t^{2m-1} + t^{2m+1})}{a + bt^m} = -z = bt^m - a \cdot \log(a + bt^m) + a \log a.$$

49. To find the integral of $(1 + 2x) \cdot \log(1 - x^3)$, assume it equals $(x + x^2) \log(1 - x^3) + z$ and differentiate; then

$$z = \int \frac{3x^3 + 3x^4}{1 - x^3}.$$

By the common rules, this quantity is not integrible in its present form; but, add to it $\frac{3x^2}{1-x^3}$ and we have $\frac{3x^2+3x^3+3x^4}{1-x^3}$, and this equals $\frac{3x^2}{1-x}$. Here, both the sum and the added quantity are separately integrible, thus:

$$\int \frac{3x^2}{1-x^3} = -\log(1-x^3).$$

$$\int \frac{3x^2}{1-x} = -\left(3x + \frac{3}{2}x^2 + 3\log(1-x)\right)$$

Therefore,

$$z = \log(1-x^3) - 3x - \tfrac{3}{2}x^2 - 3\log(1-x);$$

and

$\int (1+2x)\cdot\log(1-x^3) = (1+x+x^2)\log(1-x^3) - 3\log(1-x) - 3x - \tfrac{3}{2}x^2.$

To measure the whole area between the curve and its asymptote, that is, to ascertain the value of this integral when $x = 1$, it is necessary to determine the value of the two irrational terms, $3\log(1-x^3)$ and $-3\log(1-x)$, when $x = 1$.

In Art. 42, this was effected by making the quantity within the vinculum the same in each logarithmic term. This cannot be done in the present case; but the value may be obtained by means of the ultimate ratio; thus,

Let

$$x = 1 - a,$$

then,

$$1 - x = a$$
$$1 - x^3 = 3a - 3a^2 + a^3,$$

when x approaches to 1, a becomes infinitely small; so the ultimate ratio $1-x$ to $1-x^3$ is 1 to 3, and the difference of the logarithms of these two quantities is, therefore, log 3.

That is, $\log(1-x^3) - \log(1-x) = +\log 3$, when $x = 1$. Hence, the whole area between the curve and its asymptote is $3\log(1-x^3) - 3\log(1-x) - 3x - \tfrac{3}{2}x^2 = 3\log 3 - 4{\cdot}5$. In like manner, the integral of $(1-2x)\log(1+x^3) = (x-x^2)\log(1+x^3) - \int\frac{3x^3-3x^4}{1+x^3}$.

And

$$\frac{3x^3-3x^4}{1+x^3} + \frac{3x^5}{1+x^3} = \frac{3x^3}{1+x}.$$

Here the last two terms are separately integrible, therefore,

$\int (1-2x)\log(1+x^3) = (x-1-x^2)\log(1+x^3) + 3\log(1+x) - 3x + \tfrac{3}{2}x^2.$

And, when m is any positive whole number greater than 1, the integral of

$$(1 + 2x + 3x^2 + \cdots (m - 1)x^{m-2})\cdot\log(1 - x^m) = (1 + x + x^2 + x^3 + \cdots x^{m-1})\cdot\log(1 - x^m) - m\cdot\log(1 - x) - m(x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \cdots \frac{1}{m-1}x^{m-1});$$

and the whole area between the curve and its asymptote, that is, the value of this integral when $x = 1$, is

$$m\cdot\left(\log m - 1 - \frac{1}{2} - \frac{1}{3} - \cdots \frac{1}{m-1}\right).$$